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# WEAKLY RAMSEY P POINTS

BY

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ABSTRACT. If the continuum hypothesis (CH) holds, then for any $n$ Ramsey $P$ point $D$ and any $k > 1$ there exist many $n + k$ Ramsey $P$ points which are immediate Rudin-Keisler successors of $D$. There exist (CH) many 5 Ramsey $P$ points whose constellations are not linearly ordered.

A nonprincipal ultrafilter $D$ on $\omega$ is $n$ *Ramsey* ($n \in \omega$, $n \geqslant 1$) if $n$ is minimal for $D$ with the property that for every partition $F$: $[\omega]^2 \to n + 1$, there is a set $A \in D$ such that $F$ omits a color on $[A]^2$. $D$ is *weakly Ramsey* if $D$ is $n$ Ramsey for some $n \geqslant 1$. In **[D]**, Daguenet showed (assuming CH) that, for every $n \geqslant 1$, there exist $n$ Ramsey $P$ points which are 2 square (an $n$ Ramsey ultrafilter is 2 *square* [2 *rangé* in French] if it has $n$ nonstandard constellations). In this paper, we show (CH) that there exist many weakly Ramsey $P$ points which are not 2 square (this answers a question of Daguenet **[D]**). The underlying technique is a well-known sort of inductive construction; the specifics are in §2. §3 contains results from finite combinatorics which are used in the proofs of the main theorems in §§4 and 5. Throughout the paper, $D$ and $E$ are ultrafilters on $\omega$, and $f$, $g$, $h$ are functions from $\omega$ (or $\omega^2$) to $\omega$. $\mathbf{N}$ is the complete first order structure on $\omega$, and an element of the universe of the ultrapower $D$ prod $\mathbf{N}$ is denoted $[f]_D$. We identify $\mathbf{N}$ with its canonical image in $D$ prod $\mathbf{N}$, and if $f(E) = D$, $f^*$ denotes the canonical embedding of $D$ prod $\mathbf{N}$ into $E$ prod $\mathbf{N}$.

**1.** We assume basic results about $P$ points, the Rudin-Keisler ordering (denoted $\leqslant$), and ultrapowers of $\mathbf{N}$ and their submodels (as contained in, for example, **[P]** and **[B1]**). $E$ is a *strong immediate successor* of $D$ if $D < E$ and $(\forall D' < E)D' \leqslant D$. An $n$ *coloring* is a partition $G$: $[\omega]^2 \to n$; $G$ is *irreducible on D* if $(\forall A \in D)|G''[A]^2| = n$ ("$''$" means set image); otherwise $G$ is *reducible* on $D$. $D$ is $\leqslant n$ *Ramsey* if $D$ is principal or $D$ is $k$ Ramsey for some $k \leqslant n$; $D$ is $> n$ *Ramsey* if $D$ is not $\leqslant n$ Ramsey, and similarly for $\geqslant n$ *Ramsey* and $< n$ *Ramsey*. Then $D$ is $\leqslant n$ Ramsey iff $(\forall k \geqslant 1)$ every $(n + k)$ coloring is reducible on $D$. The following theorem consists of straightforward generalizations of results in **[B1]**, in which the term "weakly Ramsey" denotes what herein is defined as "Ramsey or 2 Ramsey".

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THEOREM 1.1. *Suppose* $f(E) = D$ *and* $E$ *is* $n$ *Ramsey. Then* $D$ *is* $\leqslant n$ *Ramsey, and* $D$ *is* $n$ *Ramsey iff* $f$ *is an* $E$*-isomorphism* (*i.e. one-to-one on a set in* $E$). *If* $D$ *is* $n-1$ *Ramsey, then* $E$ *is a strong immediate successor of* $D$.

Since an ultrafilter is minimal iff it is Ramsey, it follows that if $D$ is weakly Ramsey, then $(\exists E \leqslant D)(E$ is Ramsey). Since ZFC $\nvdash$ ($\exists$ a Ramsey ultrafilter) (see **[K]**), then ZFC $\nvdash$ ($\exists$ a weakly Ramsey ultrafilter).

Suppose that $E$ is a $P$ point, $D$ is Ramsey and $f(E) = D$. A theorem of Puritz **[P]** shows that any two nonstandard submodels of $E$ prod **N** have intersection cofinal in $E$ Prod **N**, and it follows that $f^{*\prime\prime}D$ prod **N** is included in every nonstandard submodel of $E$ prod **N**. Equivalently, $\forall g(\exists A \in E)(g$ is constant on $A$ or $f$ is $g$ fiberwise constant on $A$). It also follows that if $h$ is $f$ fiberwise one-to-one on a set in $E$, then in fact $h$ is one-to-one on a set in $E$.

THEOREM 1.2. *With* $D$, $E$, *and* $f$ *as above, let* $F\colon [\omega]^2 \to M$ $(M \in \omega)$. *Then* $(\exists A \in E)|F''\{\{x,y\} \in [A]^2\colon f(x) \neq f(y)\}| = 1$.

PROOF. Let $B_i = \{n \in \omega\colon \{m > n\colon F\{n,m\} = i\} \in E\}$ $(i = 0, 1, \ldots, M-1)$. Without loss of generality, $B_0 \in E$, and we may assume $f$ is finite-to-one on $B_0$. For each $n \in B_0$, let $A_n \in E$, $A_n \subseteq \{m \in B_0\colon m > n$ and $F\{n,m\} = 0\}$, such that $A_{n+1} \subseteq A_n$. Define $g$ by $g(n) =$ greatest $k$ such that $n \in A_k$ if $n \in A_0$, and $g(n) = 0$ if $n \notin A_0$. Then if $n \in A_0$, $g(n) = k$ implies that $k < n$ and for all $j \in B_0$, $j \leqslant k \to F\{j,n\} = 0$. It is easy to check that $g$ is not constant on any set in $E$, so there is a set $B \in E$ such that $f$ is $g$ fiberwise constant on $B$; we can assume $B \subseteq B_0$. Define a partition $G$ by

$$G\{s,t\} = \begin{cases} 0 & \text{if } (\forall z \in f^{-1}\{s\} \cap B)(\forall w \in f^{-1}\{t\} \cap B)F\{z,w\} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since $D$ is Ramsey, there is a set $S \in D$ such that $G$ assumes only one value $\eta$ on $[S]^2$. We claim $\eta = 0$ (and then the proof is complete by setting $A = B \cap f^{-1}S$). Suppose, for contradiction, that $\eta = 1$, and let $x_0$ be the least element of $X$. For each $y \in S$, $y > x_0$, there exist $n_y \in B \cap f^{-1}\{x_0\}$, $m_y \in B \cap f^{-1}\{y\}$ such that $F\{n_y, m_y\} \neq 0$. Since $B \cap f^{-1}\{x_0\}$ is finite and $S$ is infinite, $(\exists \bar{n} \in B \cap f^{-1}\{x_0\})$ $(\bar{n} = n_y$ for infinitely many $y \in S$). Since there is at most one $m_y$ in each fiber of $f$, and since $g \restriction B$ takes on different values on different fibers of $f$, there is a $\bar{y} \in S$ such that $\bar{n} = n_{\bar{y}}$, and $g(m_{\bar{y}}) > \bar{n} = n_{\bar{y}}$. But then $m_{\bar{y}} \in A_{g(m_{\bar{y}})} \subseteq A_{\bar{n}}$, so $F\{n_{\bar{y}}, m_{\bar{y}}\} = 0$, a contradiction. □

COROLLARY 1.3. *With* $D$, $E$, $f$ *as in the theorem, the range of* $f^*$ *is coinitial in the nonstandard part of* $E$ prod **N**.

PROOF. Let $[h]_E \in E$ prod $\mathbf{N} - \mathbf{N}$. Define $G\colon [\omega]^2 \to 3$ by

$$G\{x,y\} = \begin{cases} 0 & \text{if } f(x) < f(y) \text{ and } h(x) < h(y), \\ 1 & \text{if } f(x) < f(y) \text{ and } h(x) > h(y), \\ 2 & \text{if } f(x) = f(y). \end{cases}$$

By the theorem, $\exists A \in E$ such that $G$ assumes only one value $\eta$ on pairs from $A$ which lie in different fibers of $f$. Clearly $\eta \neq 2$, and it is easy to check $\eta \neq 1$, using that $[h]_E$ is nonstandard. So $\eta = 0$. Define $g$ by

$$g(n) = \begin{cases} \min\{h(k): k \in A \cap f^{-1}\{n\}\} & \text{if } n \in f''A, \\ 0 & \text{if } n \notin f''A. \end{cases}$$

Then $g$ is strictly increasing on $f''A \in D$, so $[g]_D$ is nonstandard, and $f^*([g]_D) = [g \circ f]_E \leqslant [h]_E$. □

**2.** Let $p \in {}^{\omega}\omega$ (or $p \in {}^{\omega^2}\omega$). Let $I$ be the ideal of subsets of $\omega$ (or $\omega^2$) on which $p$ is finite-to-one. A *p-fiber measure* is a map $\Gamma: I \to {}^{\omega}\omega$ satisfying:

(i) $\Gamma(X)(n) \leqslant |X \cap p^{-1}\{n\}|$,

(ii) $X \cap p^{-1}\{n\} \subseteq Y \cap p^{-1}\{n\} \to \Gamma(X)(n) \leqslant \Gamma(Y)(n)$, and

(iii) if $(\forall n \in \omega) g(n) \leqslant \Gamma(Y)(n)$, then $(\exists X \subseteq Y)\Gamma(X) = g$.

An example is the *cardinality function* $C_{X,p}$ defined by $C_{X,p}(n) = |X \cap p^{-1}\{n\}|$.

A *cut* in $D$ prod $\mathbf{N}$ is a pair $\langle S, L\rangle$ where $S \cup L = D$ prod $\mathbf{N}$, $S \cap L = \varnothing$, and $(\forall a \in S)(\forall b \in L) a < b$. Suppose $p(E) = D$ and $\Gamma$ is a $p$-fiber measure. The cut *associated to p and E via* $\Gamma$ is defined by $L = \{a \in D \text{ prod } \mathbf{N}: \exists X \in E \cap I, [\Gamma(X)]_D \leqslant a\}$, $S = D$ prod $\mathbf{N} - L$. This notion, for the case $\Gamma(X) = C_{X,p}$ (the *cardinality cut associated to p and E*) was introduced by Blass in **[B1]**; results therein generalize trivially to give

THEOREM 2.1. *Let E be a P point, $p(E) = D$, p not an E isomorphism, D nonprincipal, $\Gamma$ a p-fiber measure, and $\langle S, L\rangle$ the associated cut in D* prod $\mathbf{N}$. *Then* $\mathbf{N} \subseteq S$, $L \neq \varnothing$, *and every countable subset of L has a lower bound in L.*

Any cut in $D$ prod $\mathbf{N}$ satisfying the conclusions of Theorem 2.1 is a *fair* cut. If $a_1, a_2, a_3, \ldots$ is a strictly increasing $\omega$-sequence from $D$ prod $\mathbf{N}$, the cut defined by $S = \{b \in D \text{ prod } \mathbf{N}: (\exists n \in \omega) b \leqslant a_n\}$ is always a fair cut, since $D$ prod $(\omega; <)$ is $\aleph_1$-saturated (see **[CK**, p. 305]).

A *condition C* on a set $X$ is simply a statement about $X$; if the statement is true, we say *X satisfies C*. If $\langle S, L\rangle$ is a cut in $D$ prod $\mathbf{N}$, $\Gamma$ a $p$-fiber measure, then a set $X$ is *large* if $(\exists Y \subseteq X) Y \in I$ and $[\Gamma(Y)]_D \in L$.

THEOREM 2.2 (CH). *Let D be a P point, $\langle S, L\rangle$ a fair cut in D* prod $\mathbf{N}$, *p the first projection from $\omega^2$ to $\omega$, $\Gamma$ a p-fiber measure, $\{C_j: j \in J\}$ a set of $\leqslant \aleph_1$ conditions such that*

(i) $(\exists X \subseteq \omega^2)[\Gamma(X)]_D \in L$,

(ii) $(\forall X, A \subseteq \omega^2)[X \text{ large} \to (\exists Z \subseteq X)(Z \text{ large} \wedge (Z \subseteq A \vee Z \cap A = \varnothing))]$,

(iii) $(\forall X \subseteq \omega^2)(X \text{ large} \to (\exists Y, Z \subseteq X)(Y, Z \text{ large} \wedge Y \cap Z = \varnothing))$,

(iv) $(\forall X \subseteq \omega^2)(\forall j \in J)(X \text{ large} \to (\exists Y \subseteq X)(Y \text{ large} \wedge Y \text{ satisfies } C_j))$.

*Then there exist $2^{\aleph_1}$ P points E on $\omega^2$ such that $p(E) = D$, the associated cut in D* prod $\mathbf{N}$ (*via* $\Gamma$) *is $\langle S, L\rangle$, and for each $j \in J$, E contains a set satisfying condition $C_j$.*

PROOF. For each $Z \subseteq \omega^2$, let $C_Z$ be the condition $(X \subseteq Z \vee X \subseteq \omega^2 - Z)$. For each $f \in {}^{\omega}\omega$ such that $[f]_D \in L$, let $C_f$ be the condition $(X \in I \wedge [\Gamma(X)]_D < [f]_D)$. Let $\{D_\alpha: \alpha < \aleph_1\} = \{C_j: j \in J\} \cup \{C_Z: Z \subseteq \omega^2\} \cup \{C_f: [f]_D \in L\}$. Then the hypotheses of the theorem insure that $(\forall X$ large$)(\forall \alpha < \aleph_1)(\exists Y \subseteq X)(Y$ large and $Y$ satisfies $D_\alpha)$.

For each $\rho: \aleph_1 \to 2$ and each $\alpha < \aleph_1$, we define large sets $A_\alpha^\rho \in I$ such that $A_{\alpha+1}^\rho$ satisfies $D_\alpha$, the sequence $\{A_\alpha^\rho: \alpha < \aleph_1\}$ is almost decreasing for fixed $\rho$ (i.e. $\alpha < \beta \to A_\beta^\rho - A_\alpha^\rho$ is finite), and if $\alpha_0$ is minimal such that $\rho(\alpha_0) \neq \rho'(\alpha_0)$, then $A_\alpha^\rho = A_\alpha^{\rho'}$ for $\alpha < \alpha_0$ and $A_{\alpha_0}^\rho \cap A_{\alpha_0}^{\rho'} = \varnothing$. Then set $E_\rho = \{X \subseteq \omega^2: (\exists \alpha < \aleph_1) A_\alpha^\rho \subseteq X\}$.

The sets $A_\alpha^\rho$ are defined by induction on $\alpha$. Let $B_0 \in I$ be large (by (i)), and let $B_0^0$, $B_0^1$ be disjoint large subsets of $B_0$ (by (iii)). Set $A_0^\rho = A_0^{\rho(0)}$. If $\alpha = \gamma + 1$, let $\eta = \rho \restriction \alpha$, let $A_\beta^\eta = A_\beta^\rho$ for $\beta < \alpha$ (this is well defined by induction hypothesis), let $B_\eta^0$, $B_\eta^1$ be large disjoint subsets of $A_\gamma^\eta$, let $Y_\eta^j$ be a large subset of $B_\eta^j$ which satisfies $D_\gamma$, and let $A_\alpha^\rho = Y_{\rho\restriction\alpha}^{\rho(\alpha)}$.

For limit $\alpha$, follow the limit ordinal case in Theorem 2 in **[B1]** (with $\Gamma(X)$ in place of $C_X$) to obtain, for each $\eta: \alpha \to 2$, a large $Y_\eta$ such that $Y_\eta - A_\beta^\rho$ is finite for $\beta < \alpha$, $\rho \restriction \alpha = \eta$. Let $B_\eta^0$, $B_\eta^1$ be large disjoint subsets of $Y_\eta$, and set $A_\alpha^\rho = B_{\rho\restriction\alpha}^{\rho(\alpha)}$. □

**3.** The following discussion, up to the statement of Theorem 3.1, is adapted from **[AH]**. A *system of colors* of length $n$ is a sequence $\alpha = (\alpha_0, \ldots, \alpha_n)$ of finite, nonempty sets. An *$\alpha$ pattern $Q$* consists of a finite, linearly ordered set $X$, and a function $f_Q: \bigcup_{j \leq n}[X]^j \to \bigcup_{j \leq n} \alpha_j$ such that $f_Q''[X]^j \subseteq \alpha_j$. Let $P = (X, f_P)$ and $Q = (Y, f_Q)$ be $\alpha$ patterns. $P$ and $Q$ are *isomorphic* if there exists an order preserving bijection $\psi: X \to Y$ such that $(\forall c \in [X]^{\leq n}) f_P(c) = f_Q(\psi''c)$. If $Z \subseteq X$ and $e$, $M \in \omega$, $F: [X]^e \to M$, then $Z$ is *semihomogeneous for $F$ over $f_P$* iff $(\forall b, c \in [Z]^e)$(if $(b, f_P \restriction [b]^{\leq n}) \cong (c, f_P \restriction [c]^{\leq n})$, then $F(b) = F(c)$). $Q \rightsquigarrow (P)_M^e$ means that for any partition $F: [Y]^e \to M$, there is a $Z \subseteq Y$ such that $(Z, f_Q \restriction [Z]^{\leq n}) \cong P$ and $Z$ is semihomogeneous for $F$ over $f_Q$. In **[AH]**, Abramson and Harrington proved the following generalization of the finitary Ramsey theorem.

THEOREM 3.1. *Let $n$, $e$, $M \in \omega$, $\alpha$ a system of colors of length $n$, $P$ an $\alpha$ pattern. Then there is an $\alpha$ pattern $Q$ such that $Q \rightsquigarrow (P)_M^e$.*

COROLLARY 3.2. *Let $n, e_1, \ldots, e_k, M_1, \ldots, M_k \in \omega$, $\alpha$ and $P$ as above. Then there is an $\alpha$ pattern $Q$ such that $Q \rightsquigarrow (P)_{M_j}^{e_j}$ for all $j$, $1 \leq j < k$.*

PROOF. Let $Q_1 \rightsquigarrow (P)_{M_1}^{e_1}$, $Q_{j+1} \rightsquigarrow (Q_j)_{M_{j+1}}^{e_{j+1}}$. Then $Q_k \rightsquigarrow (P)_{M_j}^{e_j}$ for all $j$, $1 \leq j \leq k$. □

For notational simplicity, we often fail to distinguish between an $\alpha$ pattern and its underlying set; furthermore, we write $f_P \restriction Z$ rather than $f_P \restriction [Z]^{\leq n}$. Temporarily fix, for the remainder of §3, $k \in \omega$, $k \geq 1$, and let $\alpha$ be the system of colors $(1, 1, k)$. If $P = (X, f_P)$ is an $\alpha$ pattern, we view $f_P$ as a coloring on (only) the two element subsets of $X$. We say $P$ is *scattered* if for every triple $(a, b, c) \in k \times k \times k$, there are $z_1, z_2, z_3 \in X$ with $f_P\{z_1, z_2\} = a$, $f_P\{z_2, z_3\} = b$, and $f_P\{z_3, z_1\} = c$.

LEMMA 3.3. *Let $l \geqslant 3$, let $Q$ and $P = (X, f_P)$ be scattered $\alpha$ patterns such that $P \rightsquigarrow (Q)^2_l$ and $|Q| \geqslant 4$.*

(a) *If $g$ is a function with domain $X$, then $\exists Y \subseteq X$, $Y \cong Q$, such that $g$ is one-to-one or constant on $Y$.*

(b) *There exist $Y_1, Y_2 \subseteq X$, $Y_1 \cap Y_2 = \varnothing$, $Y_1 \cong Y_2 \cong Q$.*

(c) *If $A \subseteq X$, then $(\exists Y \subseteq X) Y \cong Q$ such that $Y \subseteq A$ or $Y \cap A = \varnothing$.*

(d) *If $G: [X]^2 \to l$, then $(\exists Y \subseteq X) Y \cong Q$ such that $|G''[Y]^2| \leqslant k$.*

PROOF. (a) Define $F: [X]^2 \to 2$ by

$$F\{x, y\} = \begin{cases} 0 & \text{if } g(x) = g(y), \\ 1 & \text{if } g(x) \neq g(y). \end{cases}$$

Then there is a $Y \subseteq X$, $Y = Q$, such that $(\forall x, y \in Y) F\{x, y\} = h(f_P\{x, y\})$ for a function $h: k \to 2$. It suffices to show that $|\text{range}(h)| = 1$, i.e. that $F$ is homogeneous on $Y$. Suppose not, and let $\eta_0, \eta_1 < k$ with $h(\eta_i) = i$. Find $a, b, c \in Y$ with $f_Q\{a, b\} = f_Q\{b, c\} = \eta_0$ and $f_Q\{a, c\} = \eta_1$. Then $g(a) = g(b)$ and $g(b) = g(c)$, but $g(a) \neq g(c)$, a contradiction.

(b) Let $Y_1 \subseteq X$, $Y_1 \cong Q$, and write $Y_1$ as the disjoint union of two nonempty sets $A$ and $B$. Let $C = X - Y_1$, and define $F: [X]^2 \to 3$ by

$$F\{x, y\} = \begin{cases} 0 & \text{if } \{x, y\} \subseteq A \text{ or } \{x, y\} \subseteq B, \\ 1 & \text{if } \{x, y\} \subseteq C, \\ 2 & \text{otherwise.} \end{cases}$$

Let $Y_2 \subseteq X$, $Y_2 \cong Q$, $Y_2$ semihomogeneous for $F$ over $f_P$. We have $h: k \to 3$ with $F\{x, y\} = h(f_P\{x, y\})$ for $x, y \in Y_2$; it suffices to show range$(h) = \{1\}$.

For contradiction, suppose $0 \in \text{range}(h)$, say $h(\eta) = 0$. Then $h$ must assume another value $d$, since otherwise $Y_2 \subseteq A$ or $Y_2 \subseteq B$, a contradiction since neither $A$ nor $B$ includes an isomorph of $Q$. Let $h(\rho) = d \neq 0$, and find $a, b, c \in Y_2$ with $f_P\{a, b\} = f_P\{b, c\} = \eta$ and $f_P\{a, c\} = \rho$. Thus, $a$ and $b$, as well as $b$ and $c$, are both in $A$ or both in $B$, and so $a$ and $c$ are both in $A$ or both in $B$, contradicting the choice of $a$, $c$, and $\rho$. So $0 \notin \text{range}(h)$.

Suppose now $h(\eta) = 2$ for some $\eta < k$. Then $(\exists \rho < k) h(\rho) = 1$, since any set on which $F$ assumes only the value 2 has at most three elements. Let $a, b, c \in Y_2$ with $f_P\{a, b\} = f_P\{b, c\} = \rho$ and $f_P\{a, c\} = \eta$. Then both $a$ and $c$ are in $C$, contradicting $F\{a, c\} = 2$.

(c) Apply (a) to the characteristic function of $A$.

(d) Immediate by semihomogeneity. □

## 4.

THEOREM 4.1 (CH) (BLASS [B1], FOR $n = k = 1$). *Let $D$ be an $n$ Ramsey $P$ point, $k \geqslant 1$. There exist $2^{2^{\aleph_0}} n + k$ Ramsey $P$ points $E$ which are strong immediate successors of $D$.*

PROOF. Fix $R$, a Ramsey ultrafilter, and $q \in {}^\omega\omega$ such that $q(D) = R$ (if $n = 1$, take $R = D$ and $q =$ identity). Let $\langle S_R, L_R \rangle$ be the cardinality cut in $R$ prod $\mathbf{N}$ associated to $q$ and $D$. Fix $\bar{h} \in {}^\omega\omega$ such that $[\bar{h}]_R \in L_R$, and let $H$ be an irreducible

$n$ coloring on $D$. Let $s_0 = q^*([\bar{h}]_R) = [\bar{h} \circ q]_D$, $s_{i+1} = 2^{s_i}$, and define a cut $\langle S, L\rangle$ in $D$ prod $\mathbf{N}$ by $S = \{t \in D \text{ prod } \mathbf{N}\colon (\exists i \in \omega) t \leqslant s_i\}$. Then $\langle S, L\rangle$ is a fair cut, and $S$ is closed under exponentiation, multiplication, and addition. Let $p$ be the first projection from $\omega^2$ to $\omega$.

Let $\alpha$ be the system of colors $(1, 1, k)$. Define $\alpha$ patterns $P_i'$ as follows: $P_0' = \varnothing$, $P_1'$ = any scattered $\alpha$ pattern with $|P_1'| \geqslant 4$, $P_{n+1}' \rightsquigarrow (P_n')^2_{n+k+1}$. Define $\alpha$ patterns $P_i$ by $P_0 = \varnothing$, $P_1 = P_1'$, $P_{m+1} = P_l'$, where $l$ is minimal such that

$$|P_l'| \geqslant (n + k + 1)^{\Sigma_{j<m}|P_j|}.$$

Then, for $i \geqslant 1$, $P_i$ is scattered and Lemma 3.3 holds with $P_{i+1}$ and $P_i$ in place of $P$ and $Q$, and $l = n + k + 1$.

Fix $\bar{g} \in {}^\omega\omega$ with $[\bar{g}]_D \in L$. Let $X_0 \subseteq \omega^2$ such that $|X_0 \cap p^{-1}\{m\}| = |P_{\bar{g}(m)}|$, and order $X_0 \cap p^{-1}\{m\}$ by $\langle a, m\rangle < \langle b, m\rangle$ iff $a < b$. Define $G\colon [\omega^2]^2 \to n + k$ by

(a) $(X_0 \cap p^{-1}\{m\}, G|(X_0 \cap p^{-1}\{m\})) \cong P_{\bar{g}(m)}$,

(b) if $p(x) \neq p(y)$, then $G\{x, y\} = H\{p(x), p(y)\} + k$,

(c) $G\{x, y\}$ is arbitrary if $p(x) = p(y)$ and $\{x, y\} \nsubseteq X_0$.

Define a $p$ fiber measure $\Gamma$ by $\Gamma(Y)(m) = M$, where $M$ is maximal such that $(Y \cap X_0 \cap p^{-1}\{m\}, G \restriction Y \cap X_0 \cap p^{-1}\{m\})$ includes an isomorph of $P_M$. Then $\Gamma(X_0) = \bar{g}$.

For each $f\colon \omega^2 \to \omega$, let $C_f$ be the condition on $X$: $f$ is $p$ fiberwise constant on $X$ or $(q \circ p)$ fiberwise one-to-one on $X$.

For each $F\colon [\omega^2]^2 \to n + k + 1$, $C_F$ is the condition:

$$(\forall l \in p''X)|F''[X \cap p^{-1}\{l\}]^2| \leqslant k \text{ and}$$

$$(\forall x, y, v, w \in X)[q(p(x)) = q(p(y)) = q(p(v)) = q(p(w)) \wedge p(x) = p(y) \neq p(v) = p(w) \to F\{x, v\} = F\{y, w\}].$$

We postpone the verification of the hypotheses in Theorem 2.2 and show that the $2^{2^{\aleph_0}}$ $P$ points $E$ produced by that theorem satisfy the conclusion of the present theorem. Since $E$ contains sets satisfying $C_f$ for each $f\colon \omega^2 \to \omega$, it follows by the remarks preceding Theorem 2.1 that every $f\colon \omega^2 \to \omega$ is $p$ fiberwise constant or (globally) one-to-one on a set in $E$. Hence $E$ is a strong immediate successor of $D$.

By the definition of $\Gamma$, $G$ assumes all $n + k$ values on any large set; since $E$ consists of large sets, $G$ is irreducible on $E$ and so $E$ is $\geqslant n + k$ Ramsey. Let $F$: $[\omega^2]^2 \to n + k + 1$, and let $X \in E$ satisfy $C_F$. Partition $p''X$ into $\binom{n+k+1}{k}$ pieces $W_i$ such that if $a$ and $b$ are in $W_i$, then the same $k$ colors are assumed on $[X \cap p^{-1}(a)]^2$ and on $[X \cap p^{-1}(b)]^2$. Since $p''X \in D$, one of these pieces $\overline{W}$ is in $D$; let $\eta_0, \ldots, \eta_{k-1}$ be the colors assumed on pairs in the same fiber of $Y = X \cap p^{-1}\overline{W} \in E$. By Theorem 1.2, there is a set $Y' \in E$ such that $F''\{[x, y] \in [Y']^2\colon q(p(x)) \neq q(p(y))\} = \{\rho\}$, for some $\rho < n + k + 1$. Let $Z = Y \cap Y' \in E$.

If $n = 1$, then $q$ = identity, the second half of $C_X$ is vacuous, and so $(\forall x, y \in Z)$,

$$F\{x, y\} = \begin{cases} \eta_i \text{ (some } i < k) & \text{if } p(x) = p(y), \\ \rho & \text{if } p(x) \neq p(y). \end{cases}$$

Thus $F$ assumes at most $k + 1$ colors on $[Z]^2$, so $E$ is $k + 1$ Ramsey.

If $n > 1$, define a partition $\bar{H}$: $[\omega]^2 \to n + k + 2$ by

$$\bar{H}\{l, m\} = \begin{cases} F\{x, y\} & \text{if } q(l) = q(m) \wedge x, y \in Z \wedge p(x) = l \wedge p(y) = m, \\ n + k + 1 & \text{if } q(l) \neq q(m) \vee l \notin p''Z \vee m \notin p''Z. \end{cases}$$

$\bar{H}$ is well defined since $Z$ satisfies the second half of $C_F$. Since $D$ is $n$ Ramsey, $(\exists B \in D)|\bar{H}''[B]^2| \leqslant n$, and we can assume $B \subseteq p''Z$. $\bar{H}$ must assume the value $n + k + 1$ on $[B]^2$, so $\bar{H}$ assumes at most $n - 1$ values, say $\beta_0, \ldots, \beta_{n-2}$, on $\{\{k, l\} \in [B]^2: q(k) = q(m)\}$. Let $V = Z \cap p^{-1}B \in V$. If $x, y \in V$, we have

$$F\{x, y\} = \begin{cases} \eta_i \text{ (some } i < k) & \text{if } p(x) = p(y), \\ \rho & \text{if } q(p(x)) \neq q(p(y)), \\ j \text{ (some } j < n - 1) & \text{if } q(p(x)) = q(p(y)) \wedge p(x) \neq p(y). \end{cases}$$

Thus, $E$ is $n + k$ Ramsey.

It remains to verify the hypotheses of Theorem 2.2; (i) is satisfied by the set $X_0$. For (ii), let $X$ be large, and let $A \subseteq \omega^2$; without loss of generality $X \in I$ and $[\Gamma(X)]_D \in L$. For each $l \in \omega$ such that $\Gamma(X)(l) \geqslant 1$, apply Lemma 2.2(c) to $(X \cap p^{-1}\{l\}, G \restriction X \cap p^{-1}\{l\})$ and $A \cap p^{-1}\{l\}$ to obtain $Y_l \subseteq X \cap p^{-1}\{l\}$ such that $(Y_l, G \restriction Y_l) = P_{\Gamma(X)(l)-1}$ and $Y_l \subseteq A$ or $Y_l \cap A = \varnothing$. If $\Gamma(X)(l) = 0$, let $Y_l = \varnothing$. Let $B \in D$ such that $(\forall l \in B) Y_l \subseteq A$ or $(\forall l \in B) Y_l \cap A = \varnothing$; let $Z = X \cap p^{-1}B$. Then $[\Gamma(Z)]_D = [\Gamma(X)]_D - 1 \in L$ (since $S$ is closed under the successor function), so $Z$ is large and either $Z \cap X = \varnothing$ or $Z \subseteq X$.

For (iii), let $X \in I$, $[\Gamma(X)]_D \in L$. Apply Lemma 3.3(b) in each fiber $X \in p^{-1}\{l\}$ such that $\Gamma(X)(l) \geqslant 1$ to obtain $Y_l^0$ and $Y_l^1$ with $Y_l^0 \cap Y_l^1 = \varnothing$, $Y_l^i \subseteq X \cap p^{-1}\{l\}$ and $(Y_l^i, G|Y_l^i) \cong P_{\Gamma(X)(l)-1}$. Let $Y = \bigcup Y_l^0$, $Z = \bigcup Y_l^1$.

For (iv), let $X \in I$, $[\Gamma(X)] \in L$, and let $f$: $\omega^2 \to \omega$. Apply Lemma 3.3(a) to each $p$-fiber of $X$ and argue as above to obtain a large $Y' \subseteq X$ such that $f$ is $p$ fiberwise constant or $p$ fiberwise one-to-one on $Y'$. If the former, then $Y'$ satisfies $C_f$. If the latter, let $B \in D$ such that $(\forall a \in \omega) C_{B,q}(a) \leqslant \bar{h}(a)$ (there exists such a $B \in D$ since $[\bar{h}]_R \in L_R$). Let $Y = Y' \cap p^{-1}(B)$; $Y$ is large. Temporarily fix $a \in (q \circ p)''Y$, and let $B \cap q^{-1}\{a\} = \{m_0, \ldots, m_{l-1}\}$ and $\Gamma(Y)(m_i) = M_i$. Then $l < \bar{h}(a)$.

Reorder $\{m_i: 0 \leqslant i < l\}$ so that $M_i \geqslant M_{i+1}$. Choose $\Omega_i \subseteq Y \cap p^{-1}\{m_i\}$ with $\Omega_i \cong P_{M_i - 2i}$ ($\Omega_i = \varnothing$ if $M_i - 2i \leqslant 0$). For each $i$, $0 \leqslant i \leqslant l - 2$, define $g_i$: $\Omega_i \to 2$ by $g_i(x) = 0$ iff $(\exists y \in \bigcup_{i<j<l} \Omega_j) f(x) = f(y)$. By Lemma 3.3(a), there is a set $\psi_i \subseteq \Omega_i$, $\psi_i \cong P_{M_i - 2i - 1}$, such that $g_i$ is one-to-one or constant on $\psi_i$. For nonempty $\psi_i$, $g_i$ is constant on $\psi_i$, since $|\psi_i| \geqslant 4$. For each $i$, we claim that $g_i''\psi_i = \{1\}$, since otherwise $|\psi_i| \leqslant \Sigma_{i<j<l}|\Omega_j|$; hence $|P_{M_i - 2i - 1}| \leqslant \Sigma_{i<j<l}|P_{M_i - 2j}|$, contradicting the definition of $\{P_m\}$. Let $\psi_{l-1} = \Omega_{l-1}$, and let $\Delta_a = \bigcup_{0<i<l} \psi_i$; then $f$ is one-to-one on $\Delta_a$.

Let $Z = \bigcup_{a \in (q \circ p)''Y} \Delta_a$. Then $Z$ satisfies $C_f$, and for all $m \in \omega$, $\Gamma(Z)(m) \geqslant \Gamma(Y)(m) - 2\bar{h}(q(m)) - 1$, so $[\Gamma(Z)]_D \geqslant [\Gamma(Y)] - 2[\bar{h} \circ q] - 1$. Since $[\bar{h} \circ q]_D \in S$ and $S$ is closed under addition and multiplication, it follows that $[\Gamma(Z)]_D \in L$, so $Z$ is large.

Now suppose $X \in I$, $[\Gamma(X)]_D \in L$, and $F$: $[\omega^2]^2 \to n + k + 1$. Apply Lemma 3.3(d) to each fiber $X \cap p^{-1}\{l\}$ to obtain $A_l \subseteq X \cap p^{-1}\{l\}$, $A_l \cong P_{\Gamma(X)(l)-1}$ such that $|F''[A_l]^2| \leqslant k$. Let $X' = \bigcup_{l\in\omega} A_l$; then $X'$ is large and satisfies the first half of condition $C_F$. Let $B \in D$ such that $(\forall a \in \omega)C_{B,q}(a) \leqslant \bar{h}(a)$. Let $Y = X' \cap p^{-1}B$; $Y$ is large. Fix $a \in (q \circ p)''Y$, let $B \cap q^{-1}\{a\} = \{m_0, \ldots, m_{l-1}\}$, ordered so that if $M_i = \Gamma(Y)(m_i)$, then $M_i \geqslant M_{i+1}$. Choose $\Omega_i \subseteq Y \cap p^{-1}\{m_i\}$ such that $\Omega_i \cong P_{M_i-2i}$. For each $i$, $0 \leqslant i \leqslant l-2$, define a function $g_i$ on $\Omega_i$ by $g_i(x) = \{\langle z, \eta\rangle: z \in \bigcup_{i<j<l} \Omega_j \wedge F\{x, z\} = \eta\}$. By Lemma 3.3(a), find $\psi_i \subseteq \Omega_i$, $\psi_i = P_{M_i-2i-1}$ such that $g_i$ is one-to-one or constant on $\psi_i$. We claim that $g_i$ is constant on $\psi_i$.

The number of possible functions $g_i(x)$, for $x \in \Omega_i$, is

$$(n+k+1)^{\sum_{i<j<l}|\Omega_j|} = (n+k+1)^{\sum_{i<j<l}|P_{M_j-2j}|}$$
$$\leqslant (n+k+1)^{\sum_{m<M_i-2i-1}|P_m|} < |P_{M_i-2i-1}| = |\psi_i|.$$

Thus $g_i$ is not one-to-one on $\psi_i$. Let $\psi_{l-1} = \Omega_{l-1}$; then if $i < j \leqslant l-1$, and $x, y \in \psi_i$ and $z \in \psi_j$, then $F\{x, z\} = F\{y, z\}$.

Now suppose $1 \leqslant j \leqslant l-1$ and $z \in \psi_j$. The number of possible ways to assign $n+k+1$ colors to pairs of the form $\{x, z\}$ for $x \in \bigcup_{0\leqslant i<j} \psi_i$ is $(n+k+1)^j$ (by the last sentence of the last paragraph). For each $j$, $1 \leqslant j \leqslant l-1$, partition $\psi_j$ into $(n+k+1)^j$ pieces so that if $z$ and $w$ lie in the same piece, then $(\forall x \in \bigcup_{i<j} \psi_i)F\{x, z\} = F\{x, w\}$. Apply Lemma 3.3(c) $(n+k+1)^j - 1$ times to $\psi_j$ and this partition to obtain

$$\chi_j \subseteq \psi_j, \qquad \chi_j \cong P_{M_j-2j-(n+k+1)^j}$$

such that $\chi_j$ is included in one of the pieces. Let $\chi_0 = \psi_0$.

Let $\Delta_a = \bigcup_{0\leqslant j\leqslant l-1} \chi_j$. Then $(\forall x, y, z, w \in \Delta_a)[p(x) = p(y) \neq p(z) = p(w) \rightarrow F\{x, z\} = F\{y, w\}]$. Let $Z = \bigcup_{a\in(q\circ p)''Y} \Delta_a$. Then $Z$ satisfies the second half, and hence all of, $C_F$. For all $m \in P''Z$,

$$\Gamma(Z)(m) \geqslant \Gamma(Y)(m) - 2\bar{h}(q(m)) - (n+k+1)^{\bar{h}(q(m))}.$$

Since $[h \circ q]_D \in S$ and $S$ is closed under exponentiation, $[\Gamma(Z)]_D \in L$, so $Z$ is large. □

In [**D**], Daguenet showed (CH) that for any $n$, there exist $n$ Ramsey $P$ points $E$ whose (nonstandard) constellations are linearly ordered in the form

$$\begin{array}{c} n \\ n-1 \\ \cdot \\ \cdot \\ \cdot \\ 2 \\ 1 \end{array}$$

where the number $j$ indicates that if $[f]_E$ is in that constellation, then $f(E)$ is $j$ Ramsey. Such an $E$ is said to be *strictly n Ramsey* (clearly any strictly $n$ Ramsey ultrafilter is 2 square). The main result in [**R**] is that any 2 square ultrafilter is strictly $n$ Ramsey for some $n \in \omega$. Our theorem shows (CH) that any linearly ordered, strictly increasing finite sequence which begins with 1 occurs as the constellation structure of a weakly Ramsey $P$ point. Thus, for any $k \geqslant 3$, there exist (CH) many $k$ Ramsey $P$ points which are not 2 square.

**5.** In this section we show (CH) that there exist 5 Ramsey $P$ points whose constellation structure is

$$\begin{array}{ccc} & 5 & \\ / & & \backslash \\ 2 & & 2 \\ \backslash & & / \\ & 1 & \end{array}$$

In a sense, this is the best possible result, for a result of the author [**R**] shows that for $k \leqslant 4$, any $k$ Ramsey ultrafilter has a linearly ordered constellation lattice.

We will start with any Ramsey ultrafilter $R$ and then simultaneously construct 2 Ramsey ultrafilters $D_0$ and $D_1$, with $g(D_0) = g(D_1) = R$ (for some $g \in {}^{\omega}\omega$) such that if $E$ is any ultrafilter on $\omega^2$ which extends the filter $D_0 \times D_1$ and includes the set $\{(x, y): g(x) = g(y)\}$, then $E$ will be the desired 5 Ramsey $P$ point. The construction of $D_0$ and $D_1$ follows Theorem 2.2, except that hypothesis (ii) of that theorem, which insures that the resulting filter is an ultrafilter, is not satisfied. Thus we construct a filter $F$ which then extends to the ultrafilters $D_0$ and $D_1$ (by conditions $C_Z$ below). Conditions $C_f$ force the ultrafilter $E$ to be a $P$ point, which in conjunction with condition $C_H$ will show that $E$ is 5 Ramsey.

Let $\alpha$ be the system of colors (1, 2). Thus an $\alpha$ pattern $P = (X, f_P)$ is simply a partition of $X$ into 2 pieces. We denote the ordering on $X$ by $\prec$, and let $P^i = X^i = \{x \in X: f_P\{x\} = i\}$ $(i = 0, 1)$. Let $P_0 = \varnothing$, $P_1 = \{x_1, \dots, x_8\}$ where $x_i \prec x_{i+1}$ and $P_1^0 = \{x_1, x_2, x_3, x_4\}$. Suppose we have defined $P_m$ so that

$$(*) \qquad (\forall x \in P_m^0)(\forall y \in P_m^1)x \prec y.$$

Let $P'_{m+1} \rightsquigarrow (P_m)^l_6$ for $l = 2, 3, 4$. Let $P_{m+1}$ have the same underlying set as $P'_{m+1}$, let $f_{P_{m+1}} = f_{P'_{m+1}}$, and define the ordering $\prec$ on $P_{m+1}$ from the ordering $\prec'$ on $P'_{m+1}$ by $z \prec \omega$ iff $(f_{P_{m+1}}(z) = f_{P_{m+1}}(w) \wedge z \prec' w) \vee (z \in P^0_{m+1} \wedge w \in P^1_{m+1})$. Then $P_{m+1}$ satisfies $(*)$, and it is easy to check that $P_{m+1} \rightsquigarrow (P_m)^l_6$ for $l = 2, 3, 4$.

Assume CH, let $R$ be a Ramsey ultrafilter, and let $\langle S, L\rangle = \langle \mathbf{N}, R \text{ prod } \mathbf{N} - \mathbf{N}\rangle$. Then $\langle S, L\rangle$ is a fair cut. Let $p$ be the first projection from $\omega^2$ to $\omega$, and let $\bar{X} \subseteq \omega^2$ such that $|\bar{X} \cap p^{-1}\{n\}| = |P_n|$. Define an $\alpha$ pattern $G^n$ in each fiber $p^{-1}\{n\} \cap \bar{X}$ such that $G^n \cong P_n$ (ordered by $(a, n) < (b, n)$ iff $a < b$). Define $\Gamma$ by $\Gamma(X)(n) =$ maximum $M$ such that $(X \cap \bar{X} \cap p^{-1}\{n\}, G^n \restriction X \cap \bar{X} \cap p^{-1}\{n\})$ includes an isomorph of $P_M$. For any $Y \subseteq \omega^2$, let

$$Y^i = \bigcup_{n \in \omega} \{x \in Y \cap \bar{X} \cap P^{-1}\{n\}: G^n\{x\} = i\} \qquad (i = 0, 1),$$

and let $\Delta_n Y = (Y^0 \cap p^{-1}\{n\}) \times (Y^1 \cap p^{-1}\{n\})$.

For each $H$: $[\omega^2]^l \to 6$, for $l = 2, 3, 4$, let $C_H$ be the condition: $X \subseteq \bar{X}$ and $(\forall n \in p''X)X \cap p^{-1}\{n\}$ is semihomogeneous for $H$ over $G^n$.

For each $Z \subseteq \omega^2$, $C_Z$ is the condition $(X^0 \subseteq Z \vee X^0 \cap Z = \varnothing) \wedge (X^1 \subseteq Z \vee X^1 \cap Z = \varnothing)$.

For each $f$: $\bigcup_{n \in \omega} \Delta_n X \to \omega$, $C_f$ is the condition: $X \subseteq \bar{X}$ and at least one of the following holds:

(1) $(\forall x \in X^0)(\forall y, z \in X^1)[p(x) = p(y) = p(z) \to f(x, y) = f(x, z)]$,

(2) $(\forall x, y \in X^0)(\forall z \in X^1)[p(x) = p(y) = p(z) \to f(x,y) = f(y,z)]$, or

(3) $(\forall m, n \in p''X)(m \neq n \to f''\Delta_m X \cap f''\Delta_n X = \varnothing)$.

Hypothesis (i) of Theorem 2.2 is satisfied by $\bar{X}$. (iii) can be verified easily using the following

LEMMA 5.1. *Let* $m \geqslant 1$. *Then* $(\exists Q_0, Q_1 \subseteq P_m)(Q_0 \cong Q_1 \cong P_{m-1} \wedge Q_0 \cap Q_1 = \varnothing)$.

PROOF. Similar to Lemma 3.3(b) (let $Q_0 = A \cup B$, $A \cap B = \varnothing$, $A \cap Q_0^i \neq \varnothing$, and $B \cap Q_0^i \neq \varnothing$ for $i = 0, 1$; define $F$ as in Lemma 3.3(b)). □

The parts of (iv) which demand large sets satisfying $C_H$ can be verified by arguments like those in §4; for $C_Z$, use the following

LEMMA 5.2. *If* $m \geqslant 2$, $A \subseteq P_m$, *then* $(\exists Q \subseteq P_m)[Q \cong P_{m-1} \wedge (\forall i \leqslant 1)(Q^i \subseteq A \vee Q^i \cap A = \varnothing)]$.

PROOF. Let $F$: $[P_m]^2 \to 2$ be defined by $F\{x,y\} = 1$ iff $|\{x,y\} \cap A| = 1$. Let $Q \subseteq P_m$, $Q \cong P_{m-1}$, $Q$ semihomogeneous for $F$ over $f_P$. Since $Q^i$ is homogeneous for $F$ and $|Q^i| > 3$, it follows that $F''[Q^i]^2 = \{0\}$. □

We now verify hypothesis (iv) for $C_f$.

LEMMA 5.3. *If $f$ is any function defined on* $P_m^0 \times P_m^1$, *then* $\exists Q \subseteq P_m$, $Q \cong P_{m-1}$, *such that at least one of the following holds*:

(1) $(\forall x \in Q^0)$ *$f$ is constant on* $\{x\} \times Q^1$,

(2) $(\forall x \in Q^1)$ *$f$ is constant on* $Q^0 \times \{x\}$, *or*

(3) *for each* $(x,y) \in Q^0 \times Q^1$, *$f$ is one-to-one on both* $\{x\} \times Q^1$ *and* $Q^0 \times \{y\}$.

PROOF. Define $H$: $[P_m]^3 \to 2$ by, for $x_1 \prec x_2 \prec x_3$, $H\{x_1, x_2, x_3\} = 0$ iff $[x_1, x_2 \in P_m^0 \wedge x_3 \in P_m^1 \wedge f(x_1, x_3) = f(x_2, x_3)] \vee [x_1 \in P_m^0 \wedge x_2, x_3 \in P_m^1 \wedge f(x_1, x_2) = f(x_1, x_3)]$. Let $Q \subseteq P_m$, $Q \cong P_{m-1}$, $Q$ semihomogeneous for $H$ over $f_P$. Then $H$ assumes only one value on $s_0 = \{\{x_1, x_2, x_3\} \in [Q]^3\colon x_1, x_2 \in Q^0 \wedge x_3 \in Q^1\}$ and only one value on $s_1 = \{\{x_1, x_2, x_3\} \in [Q]^3\colon x_1 \in Q^0 \wedge x_2, x_3 \in Q^1\}$. If $H''s_0 = \{0\}$, then $Q$ satisfies (2). If $H''s_1 = \{0\}$, then $Q$ satisfies (1). If $H''s_0 = H''s_1 = \{1\}$, then $Q$ satisfies (3). □

Now suppose $X$ is large and $f$: $\bigcup_{n\in\omega} \Delta_n X \to \omega$. Apply Lemma 5.3 in each fiber of $X \cap \bar{X}$ to obtain $Q_n \subseteq X \cap \bar{X} \cap p^{-1}\{n\}$, $Q_n = P_{\Gamma(x)(n)-1}$, such that $Q_n$ satisfies one of the conclusions of the lemma. Let $A \in R$ such that $(\forall n \in A)Q_n$ satisfies the same conclusion and let $Y = \bigcup_{n\in A} Q_n$. Then $Y$ is large; if the conclusion satisfied in the fibers of $Y$ is (1) or (2), then $Y$ satisfies the same conclusion in $C_f$. Assume the conclusion satisfied is (3). Define $H$: $[A]^2 \to 2$ by, for $m < n$, $H\{m, n\} = 0$ iff $\Gamma(Y)(n) \leqslant \Sigma_{j\in A; j\leqslant m}|Q_j^0| \cdot |Q_j^1| + 1$.

Since $R$ is Ramsey, there is a set $B \in R$, $B \subseteq A$, which is homogeneous for $H$. Then $H''[B]^2 = 1$, since otherwise $\{\Gamma(Y)(j)\colon j \in B\}$ is bounded, a contradiction since $Y \cap p^{-1}B$ is large.

We now define, by induction on $n \in B$, sets $\bar{Q}_n \subseteq Q_n$, $\bar{Q}_n \cong \Gamma(Y)(n) - 1$, such that if $m, n \in B$, $m < n$, then $f''(\bar{Q}_m^0 \times \bar{Q}_m^1) \cap f''(\bar{Q}_n^0 \times \bar{Q}_n^1) = \varnothing$. If $m_0$ is the least element of $B$, let $\bar{Q}_{m_0} \subseteq Q_{m_0}$, $\bar{Q}_{m_0} = P_{\Gamma(Y)(m_0)-1}$. Let $n \in B$, and suppose we have defined $\bar{Q}_m$ for $m < n$, $m \in B$. Define $\psi$: $[Q_n]^2 \to 2$ by $\psi\{x, y\} = 0$ iff $x \in Q_n^0 \wedge y \in Q_n^1 \wedge f(x, y) \notin \bigcup_{m<n; m\in B} f''\bar{Q}_m^0 \times \bar{Q}_m^1$. Let $\bar{Q}_n \subseteq Q_n$, $\bar{Q}_n \simeq P_{\Gamma(Y)(n)-1}$, $\bar{Q}_n$ semihomogeneous for $\psi$ over $f_0$. Then $\psi$ assumes only one value on $T = \{\{x, y\} \in [\bar{Q}_n]^2: x \in Q_n^0 \wedge y \in Q_n^1\}$. Since $f$ is one-to-one on $\{x\} \times \bar{Q}_n^1$, $f$ assumes at least $|\bar{Q}_n^1| \geqslant \Gamma(Y)(n) - 1 > \Sigma_{j<n; j\in B}|\bar{Q}_j^0| \cdot |\bar{Q}_j^1|$ values on $\bar{Q}_n^0 \times \bar{Q}_n^1$. Thus $\psi''T = \{0\}$; this completes the construction of $\{\bar{Q}_n: n \in B\}$.

Let $Z = \bigcup_{n\in B} \bar{Q}_n$. Then $Z$ satisfies (3) in $C_f$, and $Z$ is large, since $[\Gamma(Z)]_R + 2 \geqslant [\Gamma(X)]_R$; thus, hypothesis (iv) of Theorem 2.2 is satisfied.

Hypothesis (ii) is not satisfied by $\Gamma$, $p$, $R$, and $\langle S, L\rangle$. If we follow the proof of Theorem 2.2, omitting the steps where we satisfy (ii), then, for each $\rho$: $\aleph_1 \to 2$, we obtain a filter $F_\rho$ on $\omega^2$ (instead of an ultrafilter) such that for any condition $C_H$, $C_f$, or $C_Z$, $F_\rho$ contains a set satisfying that condition. Fix $\rho$: $\aleph_1 \to 2$, and let $F = F_\rho$. Then $F \cup \{\bar{X}^i\}$ generates an ultrafilter $\bar{D}_i$ (for $i = 0, 1$), since $F$ contains sets satisfying $C_Z$ for each $Z \subseteq \omega^2$, and $\bar{D}_i$ is a $P$ point. Since $F$ contains sets satisfying $C_H$ for all $H$: $[\omega^2]^2 \to 2$, it follows that $\bar{D}_i$ is 2 Ramsey (use Theorem 1.2).

For notational ease, we identify $\bar{X}$ with $\omega$ as follows. If $x, y \in \bar{X}$, then define $x \,\Delta\, y$ if $p(x) < p(y) \vee (p(x) = p(y) \wedge x \prec y)$, where $\prec$ is the ordering given by $G^n$. Then $\Delta$ well orders $\bar{X}$ in order type $\omega$; let $\phi$: $\bar{X} \to \omega$ be the order isomorphism. View $\bar{D}_i$ as an ultrafilter on $\bar{X}$, let $D_i = \phi(\bar{D}_i)$ and $g = p \circ \phi^{-1}$. Then $D_i$ is an ultrafilter on $\omega$, and $g(D_i) = R$ ($i = 0, 1$).

Let $p_0$ and $p_1$ be the first and second projections, respectively, from $\omega^2$ to $\omega$. By Theorem 1 in **[B2]**, there is an ultrafilter $E$ on $\omega^2$ such that $p_i(E) = D_i$ and $\{(x, y) \in \omega^2: g(x) = g(y)\} \in E$. We claim that any such $E$ is a 5 Ramsey $P$ point. First, the constellations of $E$ are not linearly ordered (con($[P_0]_E$) and con($[P_1]_E$) are incomparable), so $E$ is $\geqslant 5$ Ramsey. Since $E$ contains the set $\{(x, y): g(x) = g(y)\}$, it follows that the skies of $D_0$ prod **N** and $D_1$ prod **N** are embedded in the same sky $\bar{S}$ of $E$ prod **N** (by $p_0^*$ and $p_1^*$, respectively), which must be the highest sky of $E$ prod **N** (for a discussion of skies, see **[P]** and **[B2]**). To show that $E$ is a $P$ point, we show that $\bar{S}$ is the only sky of $E$ prod **N**.

Let $f \in {}^{(\omega^2)}\omega$. Define $\hat{f}$: $\bigcup_{n\in\omega} \Delta_n \bar{X} \to \omega$ by $\hat{f}(a, b) = f(\phi(a), \phi(b))$. Let $X \in E$ satisfy $C_{\hat{f}}$, and let $A^i = \phi''X^i$. Then $A_i \in D_i$, so $V = \{(x, y): x \in A_0, y \in A_1, g(x) = g(y)\} \in E$. Since $X$ satisfies $C_{\hat{f}}$, one of the following holds:

(1) $f$ is $p_0$ fiberwise constant on $V$,

(2) $f$ is $p_1$ fiberwise constant on $V$, or

(3) $(\forall s, t \in V)(f(s) = f(t) \to g(p_0(s)) = g(p_0(t)))$.

If (1) (resp. (2)) holds, then $[f]_E \in p_0^{*}{}'' D_0$ prod **N** (resp. $[f]_E \in p_1^{*}{}'' D_1$ prod **N**). If (3) holds, then $g \circ p_0$ is $f$ fiberwise constant on $V \in E$, so con($[g \circ p_0]_E$) lies below con($[f]_E$), and thus the sky of $[f]_E$ cannot be below the sky of $[g \circ p_0]_E$, which is $\bar{S}$. In all three cases, the sky of $[f]_E$ is $\bar{S}$ (or $[f]_E$ is standard). Thus $E$ is a $P$ point.

To show that $E$ is 5 Ramsey, let $\Sigma$: $[\omega^2]^2 \to 6$. Since $E$ is a $P$ point and $R$ is Ramsey, there is a set $V_1 \in E$ such that $\Sigma$ assumes only one value $\beta$ on $\{\{s, t\} \in [V_1]^2: g(p_0(s)) \neq g(p_0(t))\}$.

Define a partition $\Omega_3$: $[\bar{X}]^3 \to 6$ as follows. $\Omega_3\{x_1, x_2, x_3\}$ is arbitrary if

$$\neg(p(x_1) = p(x_2) = p(x_3)) \vee \{x_1, x_2, x_3\} \subseteq \bar{X}^i \quad \text{for } i = 0, 1.$$

Otherwise, for $x_1 \prec x_2 \prec x_3$, let

$$\Omega_3\{x_1, x_2, x_3\} = \begin{cases} \Sigma\{(\phi(x_1), \phi(x_2)), (\phi(x_1), \phi(x_3))\} & \text{if } x_1 \in \bar{X}^0 \wedge x_2, x_3 \in \bar{X}^1, \\ \Sigma\{(\phi(x_1), \phi(x_3)), (\phi(x_2), \phi(x_3))\} & \text{if } x_1, x_2 \in \bar{X}^0 \wedge x_3 \in \bar{X}^1. \end{cases}$$

Let $Y \in F$ satisfy $C_{\Omega_3}$. By cutting down $p''Y$ appropriately, we can assume that $\Omega_3$ takes only one value $\gamma$ (resp. $\delta$) on $\{\{x_1, x_2, x_3\} \in [Y]^3: p(x_1) = p(x_2) = p(x_3) \wedge x_1 \in Y^0 \wedge x_2, x_3 \in Y^1\}$ (resp. on $\{\{x_1, x_2, x_3\} \in [Y]^3: p(x_1) = p(x_2) = p(x_3) \wedge x_1, x_2 \in Y^0 \wedge x_3 \in Y^1\}$). Let $A_i = \phi''Y^i$ for $i = 0, 1$; then $A_i \in D_i$ and $V_2 = \{(x, y) \in \omega^2: x \in A_1, y \in A_2 \wedge g(x) = g(y)\} \in E$. We have $\Sigma''\{\{s, t\} \in [V_2]^2: p_0(s) = p_0(t)\} = \{\gamma\}$, and $\Sigma''\{\{s, t\} \in [V_2]^2: p_1(s) = p_1(t)\} = \{\delta\}$.

Define partitions $\Omega_4$ and $\bar{\Omega}_4$ on $[X]^4$ as follows. $\Omega_4$ and $\bar{\Omega}_4$ are arbitrary on $\{x_1, x_2, x_3, x_4\}$ if

$$\neg(p(x_1) = p(x_2) = p(x_3) = p(x_4)) \vee \neg|\{x_1, x_2, x_3, x_4\} \cap \bar{X}^0| = 2.$$

Otherwise, for $x_1 \prec x_2 \prec x_3 \prec x_4$, let

$$\Omega_4\{x_1, x_2, x_3, x_4\} = \Sigma\{(\phi(x_1), \phi(x_3)), (\phi(x_2), \phi(x_4))\},$$

and

$$\bar{\Omega}_4\{x_1, x_2, x_3, x_4\} = \Sigma\{(\phi(x_1), \phi(x_4)), (\phi(x_2), \phi(x_3))\}.$$

Let $Y \in F$ (resp. $\bar{Y} \in F$) satisfy $C_{\Omega_4}$ (resp. $C_{\bar{\Omega}_4}$). By cutting down $p''Y$ (resp. $p''\bar{Y}$) appropriately, we can assume that $\Omega_4$ (resp. $\bar{\Omega}_4$) assumes only one value $\eta$ (resp. $\bar{\eta}$) on $\{c \in [Y]^4: |p''c| = 1 \wedge |c \cap Y^0| = 2\}$ (resp. the same with $\bar{Y}$ for $Y$). Let $B_i = \phi''Y^i$, $\bar{B}_i = \phi''\bar{Y}^i$, $i = 0, 1$; then $B_i, \bar{B}_i \in D_i$. Let $V_3 = \{(x, y) \in \omega^2: x \in B_0, y \in B_1, g(x) = g(y)\}$, and $\bar{V}_3 = \{(x, y) \in \omega^2: x \in \bar{B}_0, y \in B_1, g(x) = g(y)\}$; then $V_3, \bar{V}_3 \in E$.

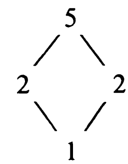
Let $W = V_1 \cap V_2 \cap V_3 \cap \bar{V}_3$. Then $W \in E \wedge (\forall s \in W) g(p_0(s)) = g(p_1(s))$. Let us agree to write an element of $[W]^2$ as $\{(x, y), (z, w)\}$ if $x < z$ or $x = z$ and $y < w$. Then, if $\{(x, y), (z, w)\} \in [W]^2$, we have

$$\Sigma\{(x, y), (z, w)\} = \begin{cases} \beta & \text{if } g(x) \neq g(z), \\ \gamma & \text{if } x = z, \\ \delta & \text{if } y = w, \\ \eta & \text{if } x < z \wedge y < w \wedge g(x) = g(z), \\ \bar{\eta} & \text{if } x < z \wedge w < y \wedge g(x) = g(z). \end{cases}$$

By our notational convention and the fact that $W \subseteq \{(x, y): g(x) = g(y)\}$, we conclude that $|\Sigma''[W]^2| \leqslant 5$.

**Theorem 5.4 (CH).** *There exist* $2^{2^{\aleph_0}}$ 5 *Ramsey P points E such that E* prod **N** *has exactly* 4 *nonstandard constellations, two of which correspond to* 2 *Ramsey ultrafilters.*

*Thus the constellation structure of these ultrafilters is*




PROOF. It only remains to show that there are no other constellations. But if there were, then $E$ would be 2 square, and then the constellations of $E$ would be linearly ordered **[R]**. □

**6.** Some natural questions follow. Does Theorem 4.1 hold if we replace "$P$ point" by "ultrafilter"? Does Theorem 5.4 hold with arbitrary $n \geqslant 6$ in the place of "5"? The ultimate problem here is to find a characterization of all those lattices which occur (CH) as the constellation structure of a weakly Ramsey ultrafilter; this problem, even restricted to $P$ points, seems quite complicated.

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